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# LAPLACIAN ENERGY FOR A BALANCED GRAPH

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**Abstract**

In this paper, we define a balanced signed Laplacian energy of a complete graph and Durer graph which is the generalised Petersen graph. Let G be a signed connected graph with order n and size m. The signed Laplacian $\bar{L}$ is defined by $\bar{L} = \bar{D} - W$, where $\bar{D}$ is a signed degree matrix and W is a symmetric matrix with zero diagonal entries. The signed Laplacian is a symmetric positive semi definite matrix. Let $\mu_1 \geq \mu_2 \geq \ldots.\mu_{n-1} \geq \mu_n = 0$ be the eigen values of the Laplacian matrix. The signed Laplacian energy is defined as $\bar{L}\,E(G) = \sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right|$. With these assumptions, we identified the Balanced Signed Laplacian energy of any graph.

**AMS Subject Classification**: 05C50, 05C69

**Key words**: Balanced signed graph, Signed Laplacian matrix, Signed Laplacian Energy of a graph

## 1 Introduction

I.Gutman and B.zhou [4,5,6,8] defined the Laplacian energy of a graph G in the year 2006. Let G be a finite connected graph with n vertices and m edges. The Laplacian matrix of the graph G, denoted by L(G) = D(G) – $A$(G) is a square matrix of order n, where D(G) is the diagonal matrix of vertex degrees of the graph G and A(G) is the adjacency matrix. Let $\mu_1, \mu_2, \ldots, \mu_n$ be the Laplacian spectrum whose Laplacian matrix G[9,10,11,12] then meet the Laplacian energy LE(G) of G which is defined as LE(G) $=\sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right|$.The weight matrix in a weighted graph is a symmetric matrix in which negative and positive entries are allowed. Such weighted graph is called signed graph. Such graph with weights -1,0,1 were introduced by Harary in 1953 and many information are cited therein the references [7,13,14,15]. Let A = $(a_{ij})$ be the adjacency matrix of the graph.The eigen values $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots\ldots\boldsymbol{\lambda}_n$, of A, assumed in the non increasing order, are the eigen values of the graph G. The energy E(G) of G is defined to be the sum of the absolute values of the eigen values of G. i.e., E(G) = $\sum_{i=1}^{n}|\lambda_i|$ [1,2,3,9].

### Basic Definitions and Examples

### Definition 1.1

A Signed graph is just an ordinary graph with each of its edges labelled with either + or a $-$.

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**Definition 1.2**

Given a signed graph G = (V, W) ( where W is a symmetric matrix with zero diagonal entries), the underlying graph of G is the graph with the vertex set V and the set of (undirected) edges E = $\{ (v_i, v_j) / w_{ij} \neq 0\}$.

**Definition 1.3**

Let G be a graph with order n and size m. The **Laplacian matrix** of the graph G is denoted by $L = (L_{ij})$ is a square matrix of order n whose elements are defined as

$$L_{ij} = \begin{cases} -1 \text{ if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 \text{ if } v_i \text{ and } v_j \text{ are not adjacent} \\ d_i \text{ if } i = j \end{cases}$$

where $d_i$ is the degree of the vertex $v_i$ .

**Definition 1.4**

If (V,W) is a signed graph where W is a (mxm) symmetric matrix with zero diagonal entries and with the other entries $w_{ij} \in R$ arbitrary. The degree of any vertex $v_i$ is defined as $\bar{d}_i = \bar{d}(v_i) = \sum_{j=1}^{m} |w_{ij}|$ and **signed degree matrix** where

$\bar{D}$ = diag ( $\bar{d}(v_1), \bar{d}(v_2), .... \bar{d}(v_m)$).

**Definition 1.5**

The **Signed Laplacian** $\bar{L}$ is defined by $\bar{L} = \bar{D} - W$, where $\bar{D}$ is signed degree matrix. The signed Laplacian is symmetric positive semi definite.

**Definition 1.6**

Let G = (V, W) be a signed graph whose underlying graph is connected. Then G is **balanced** if there is a partition of its vertex set V into two clusters $V_1$ and $V_2$ such that all the positive edges connect vertices within $V_1$ or $V_2$ and all the negative edges connect vertices between $V_1$ and $V_2$. If the signed graph has even number of negative edges then it is called a **Balanced Signed graph.**

**Definition 1.7**

Let $\mu_1, \mu_2, ..... \mu_n$ be the eigen values of $\bar{L}$ , which are called Signed Laplacian eigen values of G. The **Signed Laplacian energy** $\bar{L}E(G)$ of G is defined as $\bar{L}E(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$, where $\frac{2m}{n}$ is the average degree of the graph G.

**Example 1.8**

Signed graph is an ordered pair (G, $\sigma$), where G = (V, E) is a graph with the vertex set V and the edge set E. Let $\sigma : E \rightarrow \{p, n\}$ is a sign function, the edges with the sign p are positive and the edges with the sign n are negative. Positive edges are drawn with bold lines and the negative edges are drawn with dotted lines.

In a signed graph, if it is possible to partition the vertex set V into two clusters such that every edge that connects two vertices that belong to the same cluster is positive and every edge that connects two vertices that belong to different clusters is negative then we call the signed graph **partitionable** (or) **clusterable.**

| | $D_1$ | .......... | $D_n$ | FS |
|---|---|---|---|---|
| $S_1$ | $\tilde{c}_{11}$ | ........ | $\tilde{c}_{1n}$ | $\tilde{a}_1$ |
| ⋮ | ⋮ | | ⋮ | ⋮ |
| $S_m$ | $\tilde{c}_{m1}$ | ......... | $\tilde{c}_{mn}$ | $\tilde{a}_m$ |
| FD | $\tilde{b}_1$ | ........... | $\tilde{b}_n$ | |

**Table 1.1**

In table 1.1, let the vertex set V be partitioned into two clusters ({1,2,4,7,8}, {3,5,6,9}) in which positive edges are represented in bold lines and negative edges are denoted by dotted lines. Every connected balanced graph can be characterized as signed graph in which every cycle has an even number of negative edges. The balanced signed Laplacian matrix of the graph G is given by

$$\bar{L}(G) = \begin{pmatrix} 1 & -1 & 0 & 1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 15 & 1 & -1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 3 & 0 & -1 & -1 & 0 & 0 \\ -0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 1 \\ 0 & 1 & -1 & 1 & 6 & -1 & 0 & 1 \\ 0 & -1 & 0 & -1 & 1 & 1 & -1 & 6 \end{pmatrix}_{(8x8)}$$

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The signed Laplacian energy of the graph is $\bar{L}E(G) = \sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right|$, where the average degree of the graph is 3.56. Hence the Laplacian energy of the graph G is 20.4028 approximately.

## 2. Balanced signed Laplacian energy of a complete graph

**Example 2.1**

Consider a complete graph which has only have positive cycles. It must have either all the positive edges (or) two positive edge subgraphs $K_m$ and $K_{n-m}$ where $0 < m < n$, joined exclusively by negative edges. The complete graph is divided into two partitions. In one of the partitions, there are only two vertices and the remaining vertices are in another partition. The two partitions are connected by negative edges. The vertices are connected within the partition by positive edges. These kinds of graphs have even number of negative edges. So, they are balanced signed complete graphs.

**Example 2.2**

Consider the complete graphs $K_5$ and $K_6$ :

The characteristic equation of $\bar{L}(K_6)$ is $\mu^5 - 20\mu^4 + 150\mu^3 - 500\mu^2 + 625\mu = 0$ and the eigen values are 5, 5, 5, 5, 0. Hence the balanced signed Laplacian energy of $K_5$ is 8.

$$\bar{L}(K_6) = \begin{pmatrix} 6 & -1 & -1 & -1 & 1 & 1 \\ -1 & 6 & -1 & -1 & 1 & 1 \\ -1 & -1 & 6 & -1 & 1 & 1 \\ -1 & -1 & -1 & 6 & 1 & 1 \\ 1 & 1 & 1 & 1 & 6 & -1 \\ 1 & 1 & 1 & 1 & -1 & 6 \end{pmatrix} \quad (6X6)$$

In the same way, the characteristic equation of the signed Laplacian matrix of the $K_6$ (fig 2.2) is $10\mu^6 - 300\mu^5 + 3600\mu^4 - 21600\mu^3 + 64800\mu^2 - 77760\mu = 0$ and the eigen values are 6, 6, 6, 6, 6, 0. The balanced signed Laplacian energy of $K_6$ is 10. Consider the graph assignment problem, Minimize $Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} \times x_{ij}$

Subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i, i = 1 \text{ to } m; \sum_{i=1}^{m} x_{ij} = b_j, j = 1 \text{ to } n; x_{ij} \geq 0, i = 1 \text{ to } m \text{ and } j = 1 \text{ to } n$$

The corresponding matrix can be written as,

$$\bar{L}(K_n) = \begin{pmatrix} n-1 & -1 & -1 & \dots\dots\dots & 1 & 1 \\ -1 & n-1 & -1 & \dots\dots\dots & 1 & 1 \\ -1 & -1 & n-1 & \dots\dots\dots & 1 & 1 \\ \dots & & & & & \\ \dots & & & & & \\ 1 & 1 & 1 & \dots\dots\dots & n-1 & -1 \\ 1 & 1 & 1 & \dots\dots\dots & -1 & n-1 \end{pmatrix}_{(n \times n)}$$

Its Characteristic equation is $\mu(\mu - n)^{n-1} = 0$. Average degree of the complete graph $K_n$ is n-1. $\bar{L}E(K_n) = |0 - (n-1)| + |n - n + 1|$ (n-1) times

$$= (n-1) + 1\,(n-1)$$

$$= 2(n-1)$$

Balanced Signed Laplacian energy of a Complete graph $K_n$ is $2(n - 1)$.

## 3. Balanced signed Laplacian energy of a Durer graph

**Definition 3.1**

The Durer graph is a 3- vertex connected simple planar graph. It is one of four well-covered cubic polyhedral graphs and one of seven well- covered 3-connected cubic graphs. The only other three well-covered simple convex polyhedral are the tetrahedron, triangular prism and pentagonal prism. Durer graph is an undirected graph with 12 vertices and 18 edges and it is similar to the generalised Petersen graph.

### 3.2 Laplacian matrix of Durer graph

Durer graph is partitioned into two clusters {1,2,3,4,5,6} and {7,8,9,10,11,12}. The vertices are connected within the clusters by positive edges and the vertices are connected between different clusters by negative edges.

The eigen values of the balanced signed Laplacian matrix is 0.4216 , 0.8793 , 1.5858, 1.5858, 2.0965 , 2.6756, 3, 5.2971, 5, 4.6299, 4.4142, 4.4142

The Balanced Signed Laplacian Energy of Durer graph is $\bar{L}E(D) = 18.4143$ approximately.

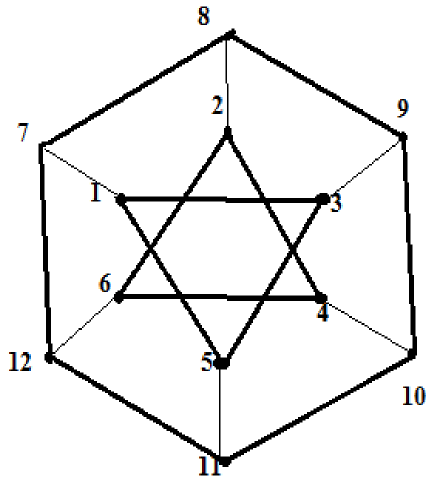


**Fig 3.1 Durer Graph**

## 4. Properties of Balanced Signed Laplacian energy of a Graph

**Property 1:** Let G be a simple, connected and balanced signed graph with n ($n\geq 2$) vertices and m edges. Let $\mu_1, \mu_2, \ldots, \mu_n$ be the Laplacian eignen values of G then the Balanced Signed Laplacian energy $\bar{L}$E(G) of G. Then $\sum_{i=1}^{n-1} \mu_i = \sum_{i=1}^{n} d_i$ = 2m and $\sum_{i=1}^{n-1} \mu_i^2 = \sum_{i=1}^{n-1} d_i^2 + \sum_{i=1}^{n} d_i = M_1 + 2m$ where $M_1$ is the sum of squares of the vertex degrees, usually referred to as the first Zagreb index.

**Property 2:** Let G be a simple, connected and balanced signed graph with n( $n\geq 2$) vertices and m edges. Then $\frac{M_1}{n} \geq 2\sqrt{\frac{M_1}{n}} \geq \frac{4m}{n}$.

**Property 3:** Let G be a simple, connected and balanced signed graph with n( $n\geq 3$) vertices and m edges. Then $\bar{L}E(G) \geq \frac{2m}{n}$.

**Property 4:** Let G be a simple, connected and balanced signed complete graph with n ($n\geq 2$) vertices and m edges. Then $\mu_1 = \mu_2 = \ldots \mu_{n-1} = n$ and $\mu_n$=0.

**Property 5:** The Balanced signed Laplacian Energy of a complete graph is never an odd integer.

**Property 6:** Let G be a simple, connected and balanced signed graph with n ($n\geq 2$) vertices and m edges. Let $\mu_1, \mu_2, \ldots, \mu_n$ be the Laplacian eignen values of G then $\gamma_i = \mu_i - \frac{2m}{n}$ ; $\sum_{i=1}^{n} \gamma_i = 0$; $\sum_{i=1}^{n} \gamma_i^2 = 2M$, where $M = m + \frac{1}{2}\sum_{i=1}^{n}( \mu_i - \frac{2m}{n})^2$. If M=m, then G is a regular graph.

**Property 7:** Let G be a simple, connected and balanced signed graph with n ($n\geq 2$) vertices and m edges then the following bounds of Balanced Signed Laplacian energy of a graph are sharp:

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i) $\bar{L}E(\mathrm{G}) \leq \sqrt{2Mn}$

ii) $\bar{L}E(\mathrm{G}) \leq \frac{2m}{n} + \sqrt{(n-1)[2M - (\frac{2m}{n})^2]}$

iii) $2\sqrt{M} \leq \bar{L}E(\mathrm{G}) \leq 2\mathrm{M}$.

## 5 Conclusion

We arrived the energy of a complete graph and Durer graph with order n and size m and identified the Balanced Signed Laplacian energy of any graph. Also it remains positive semi definite matrix with zero diagonal entries. We asserted our results with numerical examples.

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